mathematics

MDPI

Article

The Evolution of Mathematical Thinking in Chinese Mathematics Education

Na Li [1,*], Ida Ah Chee Mok [2] and Yiming Cao [3,4,*]

1 School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China
2 Faculty of Education, The University of Hong Kong, Hong Kong, China; iacmok@hku.hk
3 School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
4 International Center for Research in Mathematics Education, Beijing Normal University, Beijing 100875, China
* Correspondence: LinaCCNU@mail.ccnu.edu.cn (N.L.); caoym@bnu.edu.cn (Y.C.); Tel.: +86-10-6786-2762 (N.L.)

Received: 27 February 2019; Accepted: 20 March 2019; Published: 24 March 2019

check for updates

Abstract: Mathematical thinking (MT) has been one of the most important goals for mathematics education as it can support sustainable mathematics learning. Its role in school mathematics has recently been explicitly identified as one of "Four Basics" in the latest national curriculum standard for compulsory education, which is seen as one of the prominent features of Chinese mathematics education. This paper reviewed definitions, descriptions, and explanations from a historical perspective and aimed to provide a comprehensive and contemporary conceptualization for MT in a Chinese context, which can be applied as a comparison to MT in English works. To achieve this, document analysis was applied in this study. Major official documents, papers, and books were reviewed to see the process of MT conceptualization given by the policy makers and researchers. Results indicated that MT places more emphasis on the process of mathematical methods application in problem solving, such as the method of combination of symbolic and graphic mathematics. Mathematical thought is also recommended by Chinese researchers to help students think like mathematicians. Another major characteristic is that the classification of major types of MT is usually focused on that which can make the concept more understandable.

Keywords: mathematical thinking; mathematical method; Four Basics; curriculum standard; China

1. Introduction

Mathematical thinking (MT) is essential for teaching and learning mathematics [1]. It is not only the foundation for children' development [2] but also the cornerstone of students' sustainable development in mathematics [3]. Studies have shown that Asian mathematics teachers used to explain mathematics subjects matter in more depth and place a greater stress on thinking and understanding [4,5]. Especially in China (in this study, mainland China is considered), mathematical thinking (MT) is paid much attention in classroom instruction, which can be seen as the most important feature of Chinese mathematics education [6]. Hence, the stress on MT can help to explain Chinese students' good outstanding performance in international assessment [7–10], such as the Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA). While mathematics education in East Asian countries, such as Singapore and Japan, has been devoted significant attention, however, much less attention has been paid to Chinese mathematics education [11]. Consequently, introducing the unique characteristics in Chinese mathematics education would open a door for others to know more about Chinese mathematics education. The primary goal of this study is to provide a comprehensive understanding of mathematical thinking based on the statements of MT in the official documents.

المنارة للاستشارات
www.manaraa.com

In mainland China, a center-periphery curriculum development system is followed, and nationwide unified and mandatory curriculum standards (or syllabus) are used as guidelines for all teaching and learning activities at different grade levels and serve as a direct channel for major education reforms [12,13]. In the latest version of the *Compulsory Education Mathematics Curriculum Standard* issued in 2011 [14], one of the most important changes was the extension of "Two Basics" (here refers to basic knowledge and skills) as "Four Basics" [15]. This can be seen as a great achievement of basic education curriculum reform after entering the 21st Century in China [16]. Moreover, while "Two Basics" was considered to be a unique feature in Chinese mathematics education in the past [17,18], "Four Basics" is now seen as the unique feature of mathematics curriculum reform in China [15]. The principle of "Two Basics" was generalized to describe the characteristics of Chinese mathematics education. It literally refers to basic knowledge and basic skills [19,20]. Mathematical concepts, rules, formulas, axioms, and theorems can be included in basic knowledge, whereas, basic skills include computation, data processing and the use of calculators, simple reasoning, and drawing tables and figures [19]. Moreover, basic knowledge is mainly attained through memorization, while basic skills are primarily acquired by practice [21]. For example, manipulation on polynomial expressions, algebraic fractions, exponential and radical expressions, and memorization of the rules [22]. The final purpose was to help students solve problems, especially the examination questions, accurately and speedily [22]. For instance, when teaching the "distributive law of multiplication over addition", students are required to remember the formula "a (b + c) = ab + ac" accurately and given a number of exercises to practice and memorize the law and calculate fast. Mathematical thinking can be made more condensed and faster and reach a higher level through skillful calculations and memorization of formulas [22]. In the new curriculum standard [14], basic mathematical thinking and basic experience of mathematics activities were added to make up for "Four Basics".

From the above explanations, it can be seen that the role of MT has been in a prominent position in the latest Curriculum Standard, and not in the previous ones. Why is it that one of the "Four Basics" is in the new Curriculum Standard? To understand it, this study aimed to address the following questions: (a) how is MT developed in these official documents? (b) How are the interpretations for MT shown in these documents? (c) How are the interpretations given by researchers based on these documents?

2. Research Method

The document analysis method is " ... a systematic procedure for reviewing or evaluating documents—both printed and electronic (computer-based and Internet-transmitted material)" [23]. Researchers can apply it to their studies in order to reveal the meaning of the documents, develop understanding and discover new insights related to their research problems [24].

Specifically, researchers can understand the historical roots of specific issues and gain new visions to current phenomena by investigating the documents that bore witness to past events [23]. Documents can not only show the historical roots of specific issues, but they can also highlight specific changes and developments which can be traced by comparing various documents [23]. Documents analyses are usually broad in scope and can cover lengthy time spans while addressing, numerous events and settings [13]. Hence, the document analysis method is appropriate for the first research question, which seeks to understand the evolution of mathematical thinking from a historical perspective.

2.1. Data Selection

Document analyses require data selection rather than data collection [13]. This section describes how the documents were selected for analysis. Two aspects were considered: the Chinese key words necessary to identify materials on mathematical thinking and data resource considerations.

المنارة للاستشارات
www.manaraa.com

2.1.1. Keywords in Chinese Determined

Before looking into Chinese literature on MT, it was necessary to identify the Chinese key words relevant to it. Once these were established, this study could then discuss the meaning and evolution of mathematical thinking within a Chinese context.

Besides Sixiang being used to describe MT, there are still other relevant terms used frequently, such as Shuxuesixiangfangfa (the mathematical method of mathematical thinking/way), and Shuxuesixiang & Fangfa (mathematical thinking and method). These terms have always been used interchangeably in the literature. Considering this, these termswere selected as the key words for the purposes of document selection. The following section describes the data sources and the finalized data selection.

2.1.2. Data Sources

The literature was searched primarily through the following two ways. The first was a search of the China National Knowledge Infrastructure (CNKI) database, where almost all educational academic papers could be found. The other was a search through related books on MT. Since the 1980s, a number of important changes have taken place in the construction of Chinese databases. After nearly 30'years, three main large databases: China National Knowledge Infrastructure (CNKI), WANGFANG DATA, and CQVIP (formally known as Database Research Center under Chongqing Branch of Institute of Scientific & Technical Information of China (CB-ISTIC)) have been established and are both very popular and easy to access. Table 1 (below) provides a quick comparison of the three databases.

Table 1. Comparison of the three major Chinese databases.

Source Database	CNKI	WANGFANG	CQVIP
Academic Journals	China Academic Journals Full-text Databases (1979); China Academic Journal Network Publishing Database (1915)	China Digital Journals Full-text Database (1998) s	China science and technology Journal Database (1989)
Doctoral Dissertations & Master Theses	Chinese Selected Doctoral Dissertations and Master's Theses Full-text Databases (1984)	China Dissertation Full-text Database (1980)	N/A
Proceedings of Conference	China Proceedings of Conference Full-text Database (1953)	China Conference of Full-text Database (1985)	N/A
Newspapers	China Core Newspapers Full-text Database (2000)	N/A	China Science & Technology Economic News Database (1992)

The table reveals that CQVIP does not contain doctoral and other dissertations, conference proceedings, and journal papers focused on science and technology. WANGFANG does not collect newspapers. By contrast, CNKI is a relatively comprehensive database compared to the other two. Therefore, CNKI is the primary resource for conducting searches of Chinese literature.

2.1.3. Data Selection

With regard to official documents, the only criterion was to ensure that the documents addressed MT first. Three official documents were incorporated into this study. MT as an idea first appeared in the 1992 teaching syllabus entitled Junior Secondary School Mathematics Teaching Syllabus for Full-time Nine-year Compulsory Education [22]. The teaching syllabus has since been supplanted by the newer curriculum standards, which serve as the national documents to guide teaching and learning activities. Hence, this study included both the 2001 curriculum standard, Mathematics Curriculum Standard for Full-time Compulsory Education (Experimental Version) [25], and the more recent curriculum standard, Mathematics Curriculum Standard for Compulsory Education (2011 Version) [6].

With respect to other academic materials, the papers reviewed were selected using two primary criteria: (a) the papers should be from key journals, and; (b) if they were not from key journals, they should be cited by others frequently which can show their acceptance and popularity

المنارة للاستشارات
www.manaraa.com

among researchers. With regard to the first point, papers should be from the key journals in mathematical and general education, such as, Journal of Mathematics Education, Shuxue Tongbao, and Curriculum, Teaching Material and Method. These key journals usually have rigorous review processes, which ensure only high-quality work is published. Therefore, any papers on MT published in these journals should be reliable. Besides, this study also looked at other papers that were not published in the aforementioned key journals but were often cited by others. This may not ensure their research has been recognized and recommended by others, but it does indicate these studies have drawn a reasonable amount of attention from other scholars. Therefore, by analyzing these papers, certain insights might be found. Regarding the books, two criteria were considered: (a) important works on the study of mathematical thinking, and; (b) written by prestigious scholars in the field.

In total, three official documents, 57 academic papers and 10 books were selected and analyzed. A summary of the selected materials is presented in Table 2 (below).

Table 2. Sources of Literature Reviewed in this Study.

Key Words	Before 1978	1979–1991	1992–2000	2001–2010	2011–2019Feb	Total
Official Documents			1992 Teaching Syllabus	2001 Curriculum Standard	2011 Curriculum Standard	3
Journal Articles	1	0	8	22	26	57
Books	0	3	3	3	1	10
Total	1	3	11	25	22	70

2.2. Data Analysis

Content analysis and thematic analysis are two popular ways for analyzing documents. Content analysis refers to "the process of organizing information into categories related to the central questions of the research" [13] (p. 32). Thematic analysis is "a form of pattern recognition within the data, where emerging themes becoming the categories for analysis" [25]. Both methods require researchers to start the process with prior assumptions in their mind. However, this study seeks to gain a full understanding of mathematical thinking from a historical perspective. It becomes difficult to have some prior assumptions for this.

Hence, grounded theory formed the basic procedure for this data analysis. The researcher began with an open coding system with the aim to develop categories of concepts and themes emerging from the literature without making any prior assumptions [26,27]. Key words, terms and sentences were underlined (see Text 1).

> **Text 1**: "basic knowledge means basic concepts, principles, properties, formulas, axioms, theorems and the mathematical thinking and method reflected by mathematical content". [22] (p. 1)

Secondly, grouping similar data and giving conceptual labels to those groups could produce themes which can be used to form a conceptual scheme and ground a theory [28], for instance, the types of MT (see Text 2).

> **Text 2:** "They may have the following types. The first type is strategical thinking including thinking about transformation, . . . , etc. The second type is logical thinking including deduction, . . . etc. The third type is manual thinking including thinking about construction, discriminant of a quadric equation in one unknown, etc. . . . ". [29]

Thirdly, we reviewed and compared the key terms and/or themes in order to organize ideas and pinpoint the changes in the conception of MT over the history of Chinese mathematics education. I checked the usage of the terms MT and mathematical method stated in the official documents and found that MT was involved in mathematical methods, even in the latest *Curriculum Standard* (see below Text 3).

المنارة للاستشارات
www.manaraa.com

Text 3 MT is reflected in the process of knowledge establishment, development and application, is the abstraction and generalization of mathematical content and method in a higher level, such as abstracting, sorting, generalizing, deducting and modeling. [25] (p. 46)

After this, major characteristics were shown, and changes that happened in MT were generalized. The following section reports these findings.

3. Findings

This discussion traces evolution of mathematical thinking from the establishment of the New China following the 1949 revolution. In order to provide a thorough overview of the history of mathematical thinking in China, Figure 1 (below) provides a timeline. Key publications are presented on the left side of the vertical line while the right side records the major official documents in mathematics education and major events that have had a profound effect on mathematics education in China.

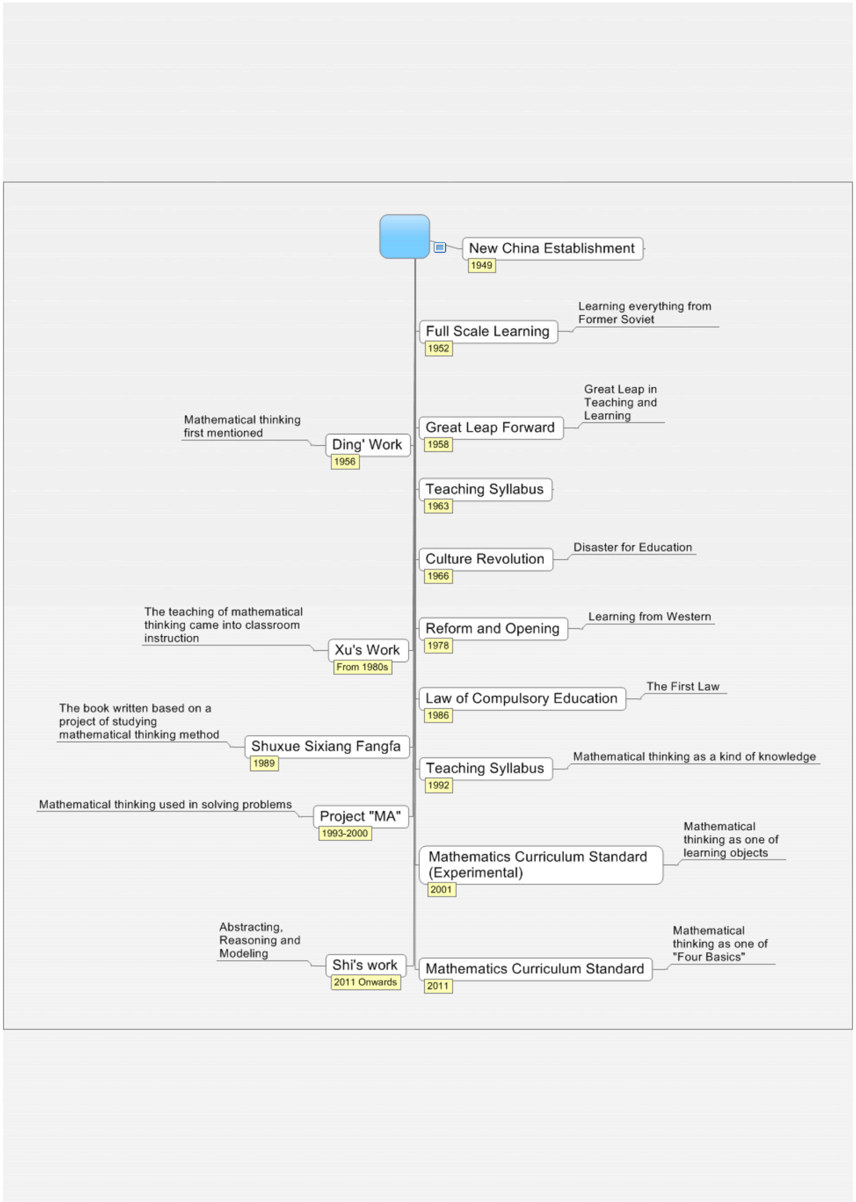

Figure 1. Overview of the evolution of mathematical thinking in Chinese mathematics education.

المنارة للاستشارات
www.manaraa.com

According to the Figure 1, only one paper of all of the publications was published before 1979. During this period, which featured the "Great Leap Forward" and the "Culture Revolution", education in China was a disaster. In 1978, China implemented an "open-door" policy and sought to realize the Four Modernizations in industry, agriculture, national defense, and science and technology, which were essential to reform in education [30]. Since the 1980s, traditional Chinese classroom teaching has been criticized by educators, because they are teacher-dominated systems of knowledge delivery that rely solely on lecture style classes, along with the corresponding passive role of the students [13]. In the meantime, a major study on MT was also started [31]. The terms mathematical thinking and methods as one of learning objectives appeared officially in the mathematics syllabus of 1992 [32], which was the most influential syllabus in Chinese mathematics education. Then, in 1999, the Ministry of Education began designing a new basic education curriculum system for the 21st century. Mathematics was officially the first school subject to be reformed [33]. Before the year 2000, the official documents issued by the central government relating to the mathematics curriculum issued were called "teaching syllabi," since then, they have been known as "curriculum standards" [34]. A mathematical thinking method appeared in the 2001 version of the *Curriculum Standard* [35]. Ten years later, another round of reforms to the curriculum standard was launched, and the latest curriculum standards were released at the end of 2011. In this version, mathematical thinking is identified as one of the "Four Basics" [14]. Overall, the role of mathematical thinking has played an increasingly important role in the history of mathematics education.

Due to the role of teaching syllabi and/or standards, the times that the official documents were issued stand as critical points. This history is summarized by four major periods; 1980–1991, 1992–2000, 2001–2010, and 2011 onwards. In addition, Ding's 1956 paper that proposed mathematical thinking and had a direct influence on all subsequent research is discussed. The main ideas of this paper are summarized and reviewed in the following section. The subsequent sections introduce the evolution of mathematical thinking over the four periods.

3.1. Earlierst Explanations Given By Ding

The article "*The development of mathematical thinking*" written by Ding Shisun [36] (Ding Shisun is a mathematician and retired politician of the People's Republic of China. He served as President of Peking University and Chairman of the China Democratic League (retrieved from https://en.wikipedia.org/wiki/Ding_Shisun, accessed on 12 December 2018). Within this one and half-page paper, he explained what could be regarded as MT. Ding defined MT as the views humans have about mathematics, such as the role of mathematics in the knowledge system, the relationship between mathematics in theory and its application, the relationship between mathematics and other sciences, and the patterns of mathematical development. All in all, three aspects in particular were considered in Ding's study of mathematical thinking. The first aspect was that every era of mathematical development could serve as a research objective. For example, what were the primary views of people on mathematics in each era? The second aspect believes it is necessary to study the research methods developed by mathematicians, past and present. There is much to learn from studying the views of notable scholars such as Descartes, Newton, Leibniz, Euler, etc., on mathematics. Thirdly, important mathematical concepts should be focused on, because these concepts reflect profound understanding of the objective world, such as number and number systems, space, sets, continuity, order, equivalence, and so on.

According to Ding, the study of MT should consider the views of people on mathematics, the development of mathematics, mathematical research methods and key mathematical concepts. As Ding stated at the beginning of the paper, the study of MT has some differences from the study of history of mathematics, but it is closely related to it. However, he did not use any references, because he asserted that no relevant studies were conducted in China. Also, he admitted that there could be relevant research in the capitalist countries (by which he meant the Western world), but he did not

المنارة للاستشارات
www.manaraa.com

know. Nevertheless, he had raised a new research topic for the future mathematics education. This can be regarded as the earliest work on the topic of MT in China.

3.2. Explanations Given by Researchers from the Persepctives of Methodology and Method (1980–1991)

During the 1980s, two main schools of thought arose, both of which were highly influential on subsequent research on MT. One was advocated by Xu Lizhi (or as L. C. Hsu): a distinguished mathematician) whose interests lay in explaining it from the perspective of mathematical methodology. The second school of thought was founded by Xie (was a professor studying dialectics of nature in Northeast China Normal University) who advocated methods for mathematical thinking.

3.2.1. Mathematical Methodology

Mathematical methodology was advocated by Xu from 1980s. He wrote a book entitled *Elective Lectures on Mathematical Methodology* and published the first version in 1983 [37]. This book was so popular among readers that it was published again five years later with only minor revisions [38]. Xu focused on ten topics to explain mathematical methodology, excluding the appendix. He began by offering a definition of mathematical methodology. According to Xu:

> Mathematical methodology is a discipline which aims to study and discuss the patterns of mathematical development, method of mathematical thinking and principles of mathematical findings, inventions and innovation. [27] (p. 1) and [28] (p. 1)

He stressed that patterns of mathematical development derived from the history of mathematics in a macro way and those specific methods and processes (such as generalizing and analogizing) in mathematics in a micro way. This was the first topic covered in the book. In the other nine chapters and the appendix, Xu introduced a number of topics he thought were relevant to mathematical methodology. However, not all of them are directly related to methods and most stressed the history of mathematics. Specifically, the three topics, along with the appendix, that illustrated methodology included: the method of mathematical modeling (chapter 2), the method of relationship-mapping-inversion (RMI) (chapter 3), the axiomatic method (chapter 4), and the appendix on the abstracting method. Of these, the RMI method proved to be the most influential among Chinese readers.

In this modeling method, a formal model can be constructed based on reality. Then, the model is analyzed to get a result and lastly the result is inverted so the problem reflects reality. The process from reality to model can be seen as a mapping and the analytical results of model can be translated back to reality by inverting the model. The principles of RMI were described as followed:

> Letting R represents a group of the relationship structures of original images and suppose the group contains x, which is an original image and to be determined. Then, letting M is a map whose function assumed that R can be mapped into relation structures of mapping image, R*, which contains the mapping image x* whose original image is x. If there was a way to determine x*, x could be known by the inversion, namely the inverse mapping, $I = M^{-1}$. [28] (p. 25)

Figure 2 (below) illustrates the process described above. R, R*, x and x* can be given the following meanings: R—the relationship system of the practical problems; R*—the relationship system of the theoretical problems; M—the process of moving from the practical problems to the theoretical problems; $I = M^{-1}$—the inverse process of moving from the theoretical problems to the practical problems; x*—the mapping image of an unknown x in a practical problem. It can be seen that modeling and abstracting are involved in this process.

المنارة للاستشارات
www.manaraa.com

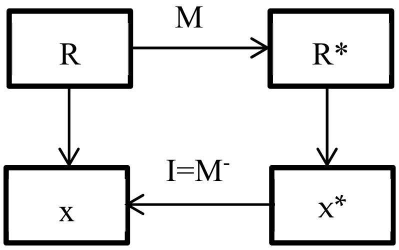

Figure 2. The relationship structure of RMI method.

The other four chapters discuss the development of other theories which are related to the history of mathematics. They are structuralism (chapter 5), polynomials solvable by radicals and Galois theory (chapter 6), nonstandard numbers and the non-Cantorian set theory in the construction of natural numbers (chapter 7), mathematics paradoxes including paradoxes and three crises in mathematical history (chapter 8) and an introduction to mathematical schools (logicism, intuitionism and formalism) (chapter 9). The last chapter had a short discussion on the mental activities, including brain storming and divergent thinking, in the invention of mathematics.

Based on the aforementioned description, even though the history of mathematics is still a key part of mathematical methodology, the author stressed the role of abstracting in mathematics. That's why he added a supplement on abstracting in the appendix in the second edition of the book. Especially in the RIM method, modeling and abstracting are indispensable.

3.2.2. Mathematical Thinking Method

To enhance the study of mathematical thinking method, a team was assembled in 1987, drawing members from 15 different provinces of China. One of major outcomes of the study group was the publication of the book Mathematical Thinking Method [39]. The study group believed that mathematical thinking should be defined on two levels.

> In the narrow sense level, mathematical thinking method refers to mathematical argumentation, operations, and thinking, methods and strategies for mathematical application. [29] (pp. 1–2)

> In the broad sense level, apart from the considerations in the narrow sense, mathematical objectives, properties, characteristics, roles, development patterns for mathematics (such as concepts, theories, methods and forms) should be included. [29] (pp. 1–2)

When compared to Xu's work, Xie and Xu placed a great deal more emphasis on methods. Concretely, they classified methods into logical methods and non-logical methods. Logical methods included: generalizing and reasoning; deductive reasoning; analogy reasoning; analysis and synthesis. At the same time non-logical methods included: image thinking; mathematical beauty and aesthetic ability; anti-mindset; and creative thinking. In addition, authors also introduced the axiomatic method, mathematical modeling method, relationship-mapping-inversion (RMI) method; constructive method; successive approximation method; duality principle method; method of mating anti-examples. Moreover, they discuss problem solving in the application of these methods. Unlike Xu who stressed abstracting but relegated it to the appendix, Xie & Xu stressed the role of conjecturing as an independent chapter.

3.3. MT and Mathematical Method Stated in the Teaching Syllabus(1992–2001)

In June 1992, the mathematics teaching syllabus for junior secondary school was released by the national education committee [22]. The document described MT and method as follows:

> The basic knowledge means basic concepts, principles, properties, formulas, axioms, theorems and the mathematical thinking and methods reflected by mathematical contents. [22] (p. 1)

المنارة للاستشارات
www.manaraa.com

This is the first time that appeared in official documents as a part of the basic knowledge which all students have to learn [40]. It needs to be noted that MT and mathematical methods are mentioned respectively. Stated this way in the official document, it wound up being a hot research topic for mathematics education researchers for the next ten years [29]. Over this period, academics argued over the definitions of the terms, the relationship between MT and method, and the types of MT.

3.3.1. Definition of MT

At the beginning of 1993, the Mathematical Accomplishment (MA) project was launched in junior secondary schools of Jiangsu province and supported by the local government. The project lasted 8 years from the beginning of 1993 to the end of 2000 and sought to develop the MT abilities of students and improve their mathematical literacy. Here, MT is eventually defined as:

> the space-forms and quantity relationships in real life reflected in human's consciousness and then processed by human's thinking, the products of thinking was mathematical thinking. [41]

They also accepted a second definition that: " . . . MT is the understanding of mathematical nature, knowledge and patterns ([32], p. 35)". The second statement was accepted and adopted by most scholars, posited that mathematical thinking is " . . . the deep understanding of mathematical nature and patterns in studying mathematics" [42] or "the deep understanding of patterns of mathematical knowledge and content development" [43].

Zang and his colleges' definition seemed like the process of abstracting or modeling, but he also stressed the profound understanding of mathematical knowledge and the patterns that were similar to others. The profound understanding of mathematics was usually interpreted as the patterns and properties within mathematics itself.

3.3.2. MT and Mathematical Method

The 1992 *Teaching Syllabus* stated that MT and methods should be put together. This made researchers and teachers discuss at great length how to distinguish between mathematical thinking and mathematical methods.

The relationship between MT and mathematical methods was also considered in the MA project. Mathematical methods are forms of MT that specifically focus on manipulative features while MT can guide the use of mathematical methods [32]. If mathematical methods are the means of construction, MT is the architectural drawings [44]. Methods are more concrete and detailed while thinking is more abstract. The MA definition of MT and methods, and the distinction that was made between the two terms were supported and cited by Cai [45] as the editor of junior secondary textbooks issued by People's Education Publisher and was used by most of junior secondary schools of China. Even the team members Sun and Zang, who reviewed more than 160 papers over a ten year period ending in 2002, yet they still hold the same beliefs as they did during the MA project [29].

Using Zhang and Guo [46] interpretations, the distinction between the two terms is clearly visible. When an important point of mathematical content, such as limits, is applied to solve a problem, it can be called method. While it is used to evaluate the meaning of itself in the mathematics system, it can be called thinking. For instance, if a "limit" is applied to find derivatives and integrals, it is called the method of limit. Conversely, if the very meaning of a limit is discussed in the process of change to be represented by numerical values and the transformation made from infinite to limited, this is called extreme thinking. If the two dimensions are combined together, it becomes known as "limit thinking".

However, earlier interpretations of mathematical methodology and mathematical method are also involved in MT. Similar to the interpretations on mathematical methodology and mathematical thinking method, Zheng [47] claimed that MT should be understood from two perspectives. The first perspective is integral to mathematics itself and focuses on specific dimensions of mathematical thought, such as limit thinking, axiomatic thinking, and sets thinking. Here, a great deal of value could

المنارة للاستشارات
www.manaraa.com

be found by taking a historical perspective. The second perspective is separable from mathematical content and addresses questions, such as how mathematicians determined their research fields and what strategies they used in problem solving. In other words, mathematical thoughts and mathematical methods should be regarded as the two major aspects for interpreting MT.

3.3.3. Types of MT

Going beyond the definition and drawing the distinction between mathematical thinking and mathematical methods, the third important finding is the ability to classify MT into typologies. This is noteworthy because both the definition for mathematical thinking and drawing the distinction between mathematical thinking and methods were not easy to clarify. Thus, by being able to classify types of MT, it may help to further clarify these ideas enough to make it possible for empirical studies.

The "MA" project provided a classification system for mathematical thinking [19,32]. The project identified three categories and 18 types of MT that were frequently used in the six junior secondary school textbooks. The results are summarized in Table 3 (below).

Table 3. Types of mathematical thinking in junior secondary education by analyzing textbooks.

Strategical	Logical	Manipulative
Thinking about transformation (*Hua Gui*)	Deducting	Construction
Abstracting & Briefly summarizing	Categorizing	Substitution
Thinking about equation & function (It means problem solving with function or equation)	Specializing	Undetermined coefficient method
Conjecturing	Analogizing	Method of completing the square
Thinking about symbolic-graphic combination	Generalizing	Parameters
Holistic & Systematic	Inversion	Discriminant

Apart from the content of the textbooks themselves, what was taught in the classroom was also considered in classification system. Zang asserted that it is essential to understand MT from the actual teaching and mathematics itself [28]. This was based on the mathematics knowledge teaching in secondary schools. In such context, the MT was usually more popular, basic and easier than other kinds of mathematical thinking, such as function thinking. This simple MT was also abstracted from concrete mathematical content and has been testified in actual life. The other was that there was a more comprehensive understanding for MT that the development of mathematics concepts, theories and methods should also be considered besides those basic MT.

3.4. Mathematical Thinking Method Stated in the Curriculum Standard (2002–2011)

In July 2001, the Ministry of Education issued *Mathematics Curriculum Standard for Compulsory Education (experimental version)* [25]. In this document, the primary learning object is that students:

> … should obtain the important mathematical knowledge including mathematical facts and activity experience, basic mathematical thinking method and basically mathematical application skills which are necessary for their future society life and further self-development. [25] (p. 3)

This is the first time that the basic "mathematical thinking method" is seen as one of the learning objects besides the "Two Basics" [48]. At the same time, this standard still failed to offer a definition or description of mathematical thinking.

3.4.1. Mathematical Thinking Method

MT is seen as being very close to mathematical methods. Over the period between 1992–2001, MT and mathematical methods had been discussed at length. The result was that in this curriculum standard, the term mathematical thinking method was used to replace both MT and methods of the

www.manaraa.com

previous teaching syllabus. In Chinese, MT and methods represent two nouns (MT& mathematical methods), while mathematical thinking method meant one single noun. In other words, what was previously two separate ideas, mathematical thinking and mathematical methods, the 2001 Curriculum Standard made into a single indivisible whole.

It was accepted that MT is the understanding of mathematical theories and methods while mathematical methods were the strategies used to solve mathematical problems [40,49]. For example, to understand the formula $(a + b)^2 = a^2+2ab+b^2$, the algebraic method (by calculating (a + b) (a + b)) can be used to explain the equation. However, if square modeling (by calculating areas of figures) was also used to explain the relationship, it would be much better to understand the formula. This method of using a combination of symbolic and graphic mathematics can help students understand mathematical formulae much better than simply using an algebraic method only. This process of learning mathematical formulae through the use combinations of numerical and graphical methods is an excellent example of mathematical thinking. There is no strict boundary between the two, so the term mathematical thinking method has been used frequently by researchers and teachers [40,49,50].

3.4.2. Major Types of MT

With the developments in mathematics education that transpired in China over this period, the importance of MT has been increasingly accepted by more and more teachers. Hence, the issue of the types of mathematical thinking remained a problem requiring further investigation [18].

MT can be classified into three levels [40]. The first level can refer to the thinking methods used to solve specific problems, such as the elimination method, the substitution method, the method of completing squares and undetermined coefficients. The second level would describe higher order logical thinking methods, for instance; analysis, synthesis, deduction, induction and analogy. The top level would include very general mathematical thinking methods including the axiomatic method and mathematical modeling. The three levels can be classified into mathematical methods, logical reasoning and mathematical modeling.

Additionally, some mathematical thinking methods have been identified as crucially important for secondary mathematical education [31]. They include letters standing for numbers indicated mathematics from constant to variable; set and corresponding; function and equation; symbolic-graphic combinations; mathematical modeling; and conversion. Whereas at the primary school level, categorization; conversion; symbolic-graphic combination; and generalization should be used in order to teach mathematical thinking method [51]. The more specific kinds of mathematical thinking became the focus in school teaching. Hence, teaching became the focal point of mathematical thinking method.

3.4.3. The Teaching of MT

The teaching of MT was discussed extensively, particularly on those points that are seen as important to secondary education, such as function, categorization and combination of symbolic-graphical thinking [38,43]. Teachers usually like to pay effort to help students to solve problems with the specific methods. Students are supposed to develop the corresponding mathematical thinking produced in applying these methods. For instance, some mathematical problems can be easily solved using functional methods. Triangles can be categorized into different types of triangles based on their properties, such as isosceles triangles and equilateral triangles. The learning of functions cannot happen without numbers and graphics, which means it usually necessary to use combination of symbolic and graphic thinking.

Moreover, there has been a significant amount of discussion over what kinds of mathematical thinking could be used to teach specific topics, such as the teaching of functions [52], plane vectors [53] and statistical knowledge [54,55]. Even the use of technology such as Geometer's Sketchpad (GSP) was considered in the teaching of mathematical thinking methods, particularly for symbolic-graphical combinations [56], because beautiful, complex and dynamic graphics can be drawn with the aid of this

المنارة للاستشارات
www.manaraa.com

software. However, in the literature, the authors were only recommending the kinds of mathematical thinking that might be appropriate, rather than anything to do with the actual teaching.

The teaching of MT can not only be implemented in secondary school level or above, rather, it needs to be implemented from primary school [57]. For example, 2 plus 1 equals 3 which is the only correct answer and the whole process can be written as 2 + 1 = 3. If the left side of "1" was changed to "2", then the right side would be from "3" to "4" correspondingly. If the left "1" became "5", then the right would be adjusted from "3" to "6". The number on the right side of the equation, namely the answer, is a unique number. As we know, in mathematics, a functional relationship is a relationship in which the value of one variable varies with changes in the value of a second variable. This can also be called an empirical or causal relationship. In such situations, the value of the right number is determined by the left two numbers, which can be seen as a function with two variables. If one of the left numbers was fixed, then the right number would be determined by the other right number, which can be seen as a function with one variable. Here, it is not necessary for the teacher to lay out the concept of a function to the pupils, but the teacher needs to have a sense of function thinking incorporated into his/her teaching [49]. Generalization, deductions, analogy, categorizations, conversions and symbolism are the main types of MT in primary school mathematics [48].

3.5. MT as One of "Four Basics"in Curriculum Standard 2011 (From 2011–2018)

3.5.1. Explanations Given for MT

To further improve the teaching and learning of mathematics, the latest version of the *Curriculum Standard for Compulsory Education Mathematics (2011 version)* [6], was released. In this document, the first learning objective is that students should " . . . obtain basic knowledge, basic skills, basic thinking and basic activity experience of mathematics which they need to survive in society and fulfill the development in the future life" [6] (p. 8). As a result, this document formalizes MT as one of the "Four Basics". This standard provides a rough definition for mathematical thinking.

> MT is reflected in the process of knowledge establishment, development and application, is the abstraction and generalization of mathematical content and method in a higher level, such as abstracting, sorting, generalizing, deducting and modeling. [6] (p. 46)

MT, rather than the mathematical thinking method, is used in the new mathematics curriculum standard. That is wise and appropriate because method could easily lead to people to think of specific methods, such as the method of deformation, observation, and comparison, rather than focusing on the thinking [58].

However, methods still serve as tools to help students develop MT. *The Standard*, for example, uses categorizing to explain the teaching aspect [58] (p. 46). Categorizing is one kind of important mathematical thinking that is frequently used by students. There are numerous ways to categorize things, for example; categorization of numbers (nature number, rational number, etc.), geometric figures, algebraic expressions, etc. Students can obtain the skill of categorizing gradually as they learn how to categorize different things through classifying the properties of each kind of items or use the method of categorization to solve problems. However, it is not easy to help students develop mathematical thinking in classroom teaching.

3.5.2. Major Types of MT

Even today, there is a great deal of debate on how to classify the types of MT. Shi developed a classification system of mathematical thinking with three main categories. Later, Gu concluded that aesthetic thinking should be a fourth category [58,59]. The categorization can be seen in Table 4.

المنارة للاستشارات
www.manaraa.com

Table 4. The types of MT under abstracting, reasoning and modeling (Shi, 2011).

Abstracting	Reasoning	Modeling	Aesthetic
Categorization	Generalization	Simplify	Concise
Set	Deduction	Quantify	Symmetry
Constant in changing	Axiomatic	Equation	Uniform
Symbolic	Number-Graph Combination	Function	Express complex through simple
Corresponding	Conversion	Optimization	Perceive the essence from appearance
Finite and infinite	Associate &Analogy	Random	
	Specialization and generalization	Statistic	

Specifically, further studies consider the types in textbooks and some specific content. Current textbooks already devote a sufficient amount of effort to developing mathematical thinking [60]. Abstracting, thinking about symbolic and graphic combination, categorizing, thinking about set, and analogizing all appear in numerical and algebra topics. On a more general level, Liu proposed three perspectives and eight types of MT [61]. The first perspective is the nature of mathematics, which includes abstracting, reasoning and modeling. The second perspective draws from the level of human thinking, incorporating categorizing, modules and thinking about sets. The third perspective is straight forward problem solving and includes two major types; thinking about symbolic and graphic combination and thinking about transformation (Huagui).

3.5.3. Teaching of MT

Attention has not only been paid to the classification systems, but also to the teaching of MT. MT is seen as the essence of mathematics, and the core of teaching mathematics should be the teaching of MT [62]. Students can have a much more robust understanding of the knowledge they have learnt, because mathematical thinking can help students develop the knowledge chains or networks in their minds [63]. Hence, the teaching of MT is being discussed far more than in previous periods [63–65].

Considering limited studies on the teaching patterns of MT in the classroom, Zhang and Fang [64] proposed a model for the teaching of MT (See Figure 3).

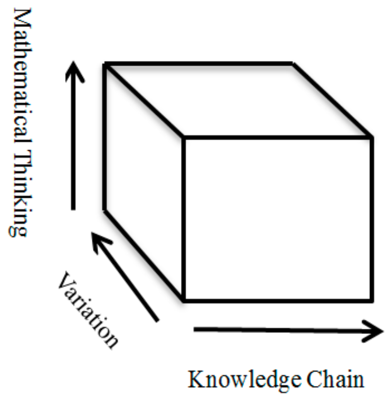

Figure 3. Model of teaching mathematical thinking.

Actually, this model incorporates three of the basics (not considering the fourth: mathematical activity experience) of Chinese mathematics education. The knowledge chain refers to the learning and grasping of basic knowledge. Variation teaching is to ask students to do exercises in order to enhance their understanding of what they have learnt through various tasks. Refining MT means that students could master the essence of knowledge. Firstly, relevant basic knowledge points can be formulated to create a knowledge chain. Then, the teacher can help establish a knowledge network through variation teaching. Taking the teaching of quadratic equation with one unknown as an example, the concept of equation, quadratic formula, and Vieta's formula could be associated in a chain. Then, with that established, various examples, exercises and test items could be provided to solidify that knowledge.

www.manaraa.com

At the same time, mathematical thinking skills, such as numerical-graphical combinations could be used in teaching the topic.

It should be noted that this model is constructed based on one knowledge chunk not one lesson. In one lesson, it is difficult for students to recall all of the related knowledge to this lesson and to develop a knowledge chain or network in one lesson. This means that this model is difficult for teachers to use in their teaching, especially in situations where student academic performance is not so good.

3.6. Conclusions and Discussions

3.6.1. The Struggling Relationship between MT and Mathematical Methods (methodology)

Mathematical method has always been considered in mathematical thinking. The relationship between mathematical thinking and methods was also discussed. It can easily be seen from the terminology used in the three official documents, mathematical thinking and method in *Teaching Syllabus 1992* [22], mathematical thinking method in Curriculum Standard 2001 [25] and mathematical thinking in the latest Curriculum Standard 2011 [6]. The role of mathematical thinking has evolved from being a kind of knowledge to a place of prominence as one of the four basics. It would seem that the role of methods has become weaker but, in reality, that weakening is just superficial.

These terms have always been used interchangeably. Sometimes, it seems that there is no appreciable difference between the meaning of mathematical thinking and mathematical thinking method in mainland China [39]. Sometimes, it is explained that when mathematical thinking is used to a solve problem, and after refining a certain type of problem, a set of proficient procedures will gradually be formed, which then become known as mathematical methods [58]. Until 2010, there were still no acceptable and common definitions that will unequivocally separate the two terms [48]. The definition given by the Curriculum Standard 2011 explains the deep-thinking ways (abstracting and generalizing) of mathematical methods is one aspect of mathematical thinking. This seems still not easy understanding by researchers and teachers.

From a historical perspective, Ding's work offered a starting point to the study of mathematical thinking in China. The perspectives that humans have about mathematics are seen as mathematical thinking [26]. However, that was far too broad to understand mathematical thinking. Xu Xie combined mathematical method to mathematical thinking which [28,29] laid a foundation for mathematical thinking method. Their work has led subsequent researchers and teachers to struggle with methods in mathematical thinking in China.

Thus, both researchers and teachers have found trouble distinguishing between mathematical thinking and mathematical methods. The two combined together become the mathematical thinking method, which is frequently used by teachers and researchers in their respective professions [66]. Moreover, in terms of teaching, methods receive much more attention than thinking. Teachers may just care about whether students can use specific methods to solve problems rather than whether or not they understand the method and why they should choose it. In such cases, teachers may not be aware what kinds of mathematical thinking would enhance their teaching.

3.6.2. Involvement in Mathematical Thought Process

When mathematical thinking appeared for the first time in Ding's paper, the development of mathematics including major concepts and methods used by mathematicians were included [26]. Since then, the development of mathematics has frequently incorporated elements of mathematical thinking [29,34,38]. The reason for this is likely the result of the introduction of new books from the outside world. One example would be the introduction of Morris Kline's classic work Mathematical thought from ancient to modern times (Vol. 1–4) [67–70]. These books introduced classical mathematical thought, which included the history of mathematics and made Chinese researchers to explain mathematical thinking from this way a second influential book would be How to read and do proofs:

المنارة للاستشارات
www.manaraa.com

An introduction to mathematical thought process [71]. This book had a profound effect on Chinese mathematics education. The following methods are recommended by this book: construction method, choose method, specialization, nested quantifiers, contradiction method, contrapositive method, induction, either/or methods, max/min methods. Although these methods can be used in problem solving, the book's value lies in its emphasis on the process of these methods. All of these books have had a broad effect on Chinese mathematics education [72].

Actually, the first emphasizes the history of mathematics while the latter focuses on the process of applying mathematical methods to solve problems. Overall, even though the history of mathematics was a main component in considering mathematical thinking, it did not appear in the latest *Curriculum Standard* for compulsory education. The history of mathematics has been an independent aspect for teachers to teach and for students to learn mathematics.

3.6.3. Focus on the Classification of MT

In addition to mathematical methods and the history of mathematics, another characteristic of mathematical thinking in Chinese mathematics education is seeking the classification of mathematical thinking. The types or levels of mathematical thinking were classified from the perspective of problem solving [32,40]. Considering the complexity of mathematical thinking and the future of mathematics education, Shi [73] asserts that all types of mathematical thinking can be classified into abstracting, reasoning and modeling.

Two possible reasons can provide a slice sight for this phenomenon of mathematical thinking classification. One is that this can help in understanding the nature of mathematical thinking given the lack of a clear definition of mathematical thinking. Specific kinds of mathematical thinking can provide an outline of the broader concept of mathematical thinking. The other is for teaching. Although the role of mathematical thinking has been established in the official documents, it is not easy for teachers to implement. The classification system, particularly with regard to the types of mathematical thinking can be taught in secondary education, can make the lonely term of mathematical thinking imaginable.

In general, the classification system is a benefit to the understanding of mathematical thinking, especially for teachers. Compared to teaching in the United States, East Asian teachers are more likely to use strategies that facilitate constructive thinking and conceptual understanding of mathematics in their students than American teachers [4,74]. However, due to the mathematical methods involved mathematical thinking, teachers tend to teach specific methods to help students solve problems faster and get good grades in examinations.

It is necessary to have a common goal for all educational research to advance the theory of learning and enhance the practice of teaching [75]. This study can broaden the understanding of mathematical thinking through Chinese policy perspective. It should be noted that this study did not attempt to identify the exact definition of mathematical thinking. Rather, it aimed to provide a comprehensive understanding on mathematical thinking within a Chinese context. Besides the theoretical contribution, the results also make the comparison of mathematical thinking between China and other cultures achievable in the future.

Author Contributions: Conceptualization, N.L.; Method, N.L, and A.C.M; Formal Analysis, N.L., A.C.M, and Y.C.; Investigation, N.L.; Recourses, Y.C.; Writing-Original Draft Preparation, N.L.; Writing-Review &Editing, N.L., A.C.M, and Y.C.; Supervision, A.C.M, and Y.C.; Project Administration, Y.C.; Funding Acquisition, Y.C.

Funding: This research was funded by self-determined research funds of CCNU from the colleges' basic research and operation of MOE' (Grant No. CCNU19A03005&CCNU19TD023) and supported by China National Education Science Grant (2018-2020): Middle School Students' Cognitive interaction and Social interaction in Collaborative Problem Solving (Grant No. BHA 180157).

Acknowledgments: This paper's editing was financially supported by China National Natural Science Foundation (2019) "Mathematical Model and Application for Target Teaching in Big Data Environment" (Grant No. 61877023).

Conflicts of Interest: The authors declare no conflict of interest.

المنارة للاستشارات

www.manaraa.com

References

1. Stacey, K. (Ed.) What is mathematical thinking and why is it important. In Proceedings of the APEC-Tsukuba International Conference 2007: Innovative Teaching of Mathematics Through Lesson Study(ii), Focusing on mathematical Thinking, Tsukuba, Japan, 2–7 December 2006.
2. Papadakis, S.; Kalogiannakis, M.; Zaranis, N. Improving mathematics teaching in kindergarten with Realistic Mathematical Education. *Early Childhood Educ. J.* **2016**, *45*, 369–378. [CrossRef]
3. Dong, L.; Clarke, D.; Cao, Y.; Wang, L.; Seah, WT. Teacher Questioning Practices over a Sequence of Consecutive Lessons: A Case Study of Two Mathematics Teachers. *Sustainability* **2019**, *11*, 139. [CrossRef]
4. Stigler, J.W.; Perry, M. Cross cultural studies of mathematics teaching and learning: Recent findings and new directions. In *Perspectives on Research on Effective Mathematics Teaching*; Grouws, D., Cooney, T., Eds.; Lawrence Erlbaum Associates Inc.: Hillsdale, NJ, USA, 1988; pp. 194–223.
5. Stigler, J.W. Research into practice: The use of verbal explanation in Japanese and American classrooms. *Arith. Teach.* **1988**, *36*, 27–29.
6. Zhang, D. The features of mathematics education in China (In Chinese). *People's Education* **2010**, *1*, 36–38.
7. Cai, J.; Nie, B. Problem solving in Chinese mathematics education: Research and practice. *Zentralblatt für Didaktik der Mathematik (ZDM)* **2007**, *39*, 459–473. [CrossRef]
8. Lapointe, A.E.; Mead, N.A.; Phillips, G.W. *A World of Difference: An International Assessment of Mathematics and SCience*; Educational Testing Service: Princeton, NJ, USA, 1989.
9. PISA 2012 Results in Focus: What 15-year-olds know and what they can do with what they know [Internet]. 2014. Available online: http://www.oecd.org/pisa/keyfindings/pisa-2012-results-overview.pdf (accessed on 26 August 2014).
10. Leung, K.S.F. IEA Studies in Mathematics and Science. *Int. Ency. Educ.* **2010**, *4*, 650–655.
11. Schoenfeld, A.H. *Foreword. How Chinese Teach Mathematics and Improving Teaching*; Routledge: New York, NY, USA, 2013; pp. xi–xv.
12. Li, Y.; Zhang, J.; Ma, T. School mathematics textbook design and development practices in China. In *Mathematics Curriculum in School Education*; Li, Y., Lappan, G., Eds.; Springer: Dordrecht, The Netherland, 2014; pp. 305–326.
13. Liu, J.; Li, Y. Mathematics curriculum reform in the Chinese mainland: Changes and challenges. In *Reforms and Issues in School Mathematics in East Asia: Sharing and Understadning Mathematics Education Policies and Practices*; Leung, F.K.S., Li, Y., Eds.; Sense Publishers: Rotterdam, The Netherlands, 2010; pp. 9–31.
14. Ministry of Education PRC. *Mathematics Curriculum Standard for Compulsory Education (2011 version)*; Beijing Normal University Publishing Group: Beijing, China, 2012. (In Chinese)
15. Sun, X. Four basics: The most important achievement of mathematics currculum reforms in the recent decade. *Basic Educ. Course J.* **2011**, *Z2*, 34–37. (In Chinese)
16. Cui, Y.; Kong, F. The analysis and strategies for the issues in implementing "Four Basic". *J. Chin. Soc. Educ.* **2014**, *3*, 53–57. (In Chinese)
17. Shi, N. The basic mathematical thinking. *Shuxue Tongbao* **2011**, *50*, 1–9. (In Chinese)
18. Zhu, C. Discussions on the teaching of mathematical thinking method. *Shuxuetongxun* **2004**, *9*, 5–7. (In Chinese)
19. Tang, H.; Peng, A.; Cheng, B.; Kuang, K.; Song, N. Characteristics of "Two Basics" teaching in secondary mathematics classrooms in China. In *How Chinese Teach Mathematics and Improve Teaching*; Li, Y., Huang, R., Eds.; Routledge: New York, NY, USA, 2013.
20. Ding, R.; Wong, N.Y. Mathematics curriculum reform in China: Latest development and challenges. In *Curriculum Reform in China*; Yin, H.-B., Lee, J.C.-K., Eds.; Nova Science Publishers, Inc.: New York, NY, USA, 2012; pp. 81–93.
21. Shao, G.; Fan, Y.; Huang, R.; Ding, E.; Li, Y. Mathematics classroom instruction in China viewed from a historical perspective. In *How Chinese Teach Mathematics and Improve Teaching*; Li, Y., Huang, R., Eds.; Routledge: New York, NY, USA, 2013; pp. 11–28.
22. Zhang, D.; Li, S.; Tang, R. The "Two Basics": Mathematics teaching and learning in mainland China. In *How Chinese Learn Mathematics: Perspectives from Insiders. 1*; Lianghuo, F., Ngai-Ying, W., Cai, J., Li, S., Eds.; World Scientific Publishing: Singapore, 2004; pp. 189–201.
23. Bowen, G.A. Document analysis as a qualitative research method. *Qual. Res. J.* **2009**, *9*, 27–40. [CrossRef]

المنارة للاستشارات

www.manaraa.com

24. Merriam, S.B. *Case Study Research in Education: A Qualitative Approach*; Jossey-Bass: San Francisco, CA, USA, 1988.
25. Fereday, J.; Muir-Cochrane, E. Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *Int. J. Qual. Methods* **2006**, *5*, 80–92. [CrossRef]
26. Huberman, M.; Miles, M.B. *The Qualitative Researcher's Companion*; Sage: Thousand Oaks, CA, USA, 2002.
27. Strauss, A.; Corbin, J.M. *Grounded Theory in Practic*; SAGE Publications, Inc.: London, UK, 1997.
28. Strauss, A.; Corbin, J. *Basics of Qualitative Research: Grounded Theory and Procedures and Techniques*; SAGE Publications, Inc.: Thousand Oaks, CA, USA, 1990.
29. Sun, C.; Zang, L. A literature review on mathematical thinking. *Maths Teach. Middle Sch.* **2002**, *10*, 28–30. (In Chinese)
30. Xu, B. The development of school mathematics textbooks in China since 1950. *Zentralblatt für Didaktik der Mathematik (ZDM)* **2013**, *45*, 725–736. [CrossRef]
31. Li, Y. Lecture on mathematical thinking and mathematical method 2009. Available online: http://mingshi.qlteacher.com/studio/xuguangmin/Article/12816 (accessed on 13 December 2018).
32. Ministry of Education PRC. *Junior Secondary School Mathematics Teaching Syllabus for Full-Time Nine-Year Compulsory Education*; People's Education Press: Beijing, China, 1992. (In Chinses)
33. Sun, X. Mathematics curriculum standards of China: Its process, strategies, outcomes and difficulties. In *Mathematics Curriculum in Pacific rim Countries—China, Japan, Korea, and Singapore*; Usisikin, Z., Willmore, E., Eds.; Information Age Publishing, Inc.: Charlotte, NC, USA, 2008; pp. 73–82.
34. Lv, S.; Ye, P. The features and implications of curriculum objects for secondary school mathematics in China from the year of 1949. *Shuxue Jiaoxue Yanjiu* **2013**, *32*, 2–9. (In Chinese)
35. Ministry of Education PRC. *Mathematics Curriculum Standard for Full-Time Compulsory Education (Experimental vrsion)*; Beijing Normal University Publishing Group: Beijing, China, 2001. (In Chinese)
36. Ding, S. The development of mathematical thinking (In Chinese). *J. Dialectics Nat.* **1956**, 15–16.
37. Xu, L. Topics in Mathematical Methodology. In *Topics in Mathematical Methodology*; Hua Zhong Gong Xue Yuan Chu Ban She: Wuhan, China, 1983. (In Chinese)
38. Xu, L. *Topics in Mathematical Methodology (2nd version)*; Hua Zhong Gong Xue Yuan Chu Ban She: Wuhan, China, 1988. (In Chinese)
39. Xie, E.; Xu, B. (Eds.) *Mathematical Thinking Method*; Shandong Education Press: Jinan, China, 1989. (In Chinese)
40. Luo, Z. The teaching of mathematical thinking method. *Zhongxuejiaoyan* **2004**, *7*, 28–33. (In Chinese)
41. Zang, L. The report of the teaching experiment "Developing students' mathematical thinking & Imporving students' mathematical literacy". *Curiculum Teach. Mater. Method* **1997**, *8*, 35–39. (In Chinese)
42. Qu, l. Disscussion on mathematical thinking. *J. Math. Educ.* **1994**, *3*, 32–35.
43. Zhang, J. Form, content, mathematical thinking method and the teaching process. *Shuxue Tongbao* **1998**, *11*, 2–4. (In Chinese)
44. Zang, L. An analysis of mathematical thinking and its features. *Maths Teach. Middle Sch.* **1998**, *5*, 1–2. (In Chinese)
45. Cai, S. Mathematical thinking and mathematical method. *Zhongxueshuxue* **1997**, *9*, 1–4. (In Chinese)
46. Zhang, D.; Guo, B. *The Draft of Mathematical Methodology*; Shanghai Education Publisher: Shanghai, China, 1996. (In Chinese)
47. Zheng, Y. Mathematical thinking, mathematical thinking method and mathematical methodology. *Sci. Technol. Dialectics* **1993**, *10*, 1–7. (In Chinese)
48. Wang, L. Practice and discussion on penetrating mathematical thinking method in primary school. *Curiculum Teach. Mater. Method* **2010**, *30*, 53–58. (In Chinese)
49. Guo, L.; Chen, Y. Discussion on the educational value of mathematical thinking method. *Theory Pract. Educ.* **2005**, *25*, 59–60. (In Chinese)
50. Liu, X. The role of mathematical thinking method in mathematics education. *J. Cap. Norm. Univ. (Soc. Sci. Ed.)* **2001**, *2*, 115–119. (In Chinese)
51. Chen, Z. The teaching design of "Squared difference formula" presentation in the project of "Core concepts, theory and ways of thinking in teaching design". *J. Chin. Math. Educ.* **2010**, *7–8*, 6–8. (In Chinese)
52. Zhang, J.; Tao, W. Emphasizing the students' thinking in learning the concept of function. *Shuxue Tongbao* **2009**, *48*, 19–25. (In Chinese)

www.manaraa.com

53. Xu, T.; Cheng, Z. Discussion on mathematical thinking method in the teaching of plane vector. *Bull. Math. (Wuhan)* **2001**, *3*, 22–23. (In Chinese)
54. Wei, X.; Jiang, G. Mathematical thinking in probability and statistics content. *J. Shaanxi Inst. Educ.* **2003**, *19*, 67–69. (In Chinese)
55. Zhong, Z.; Ning, L.; Bai, H. Discussion on the teaching tactics of mathematical thinking method. *J. Math. Educ.* **2007**, *16*, 13–16. (In Chinese)
56. Hu, J. Discussion on the teaching of mathematical thinking method in a penetrating way with the help of GSP. *Bull. Math.* **2003**, *1*, 16–17. (In Chinese)
57. Zhang, J. Sensing the power of mathematical thinking in primary school teaching for primary school mathemtics teachers. *Renminjiaoyu* **2007**, *18*, 32–35. (In Chinese)
58. Gu, P. How "two basics" developed to "four basics" in mathematic education. *J. Math. Educ.* **2012**, *21*, 14–16. (In Chinese)
59. Gu, P. Mathematical thinking also emphasized in primary school mathematics. Xiaoxuejiaoxue(Shuxue) 2012, 7–8, 14–18. *Xiaoxuejiaoxue(Shuxue)* **2012**, *7–8*, 14–18. (In Chinese)
60. Liang, Q. Strategies for helping students develop matheamtical thinking. *Curiculum Teach. Mater. Method* **2015**, *35*, 54–58. (In Chinese)
61. Liu, J. Major types of mathematical thinking and their meaning in primary school mathematics. *Curiculum Teach. Mater. Method* **2015**, *35*, 49–53. (In Chinese)
62. Ma, Z. Emphasis on the teaching of mathematical thinking and deloping students mathematical thinking: An interview of Professor Qi Chunxia. *Teacher's J.* **2013**, 3–8. (In Chinese)
63. Li, H. Emphasis on the teaching of mathematical thinking method— " Core mathematical concepts and the structure of mathematical thinking and their teaching in theory and practice in middle school" (Summary of the sixth meeting for the topic). *J. Chinese Math. Educ.* **2011**, *1–2*, 11–13. (In Chinese)
64. Zhang, D.; Fang, J. On the teaching of mathematical thinking method. *Mon. J. High Sch. Math.* **2012**, *6*, 1–3. (In Chinese)
65. Gu, P. How to teach mathematical thinking in compulsory education phase. *Jichujiaoyukecheng* **2012**, *103*, 35–40. (In Chinese)
66. Cheng, H. Reflection on the teaching of mathematical thinking method for secondary mathematics. *Shuxue Tongbao* **2012**, *51*, 28–32. (In Chinese)
67. Kline, M. *Mathematical Thought from Ancient to Modern Times (1)*; Shanghai Science and Technology Press: Shanghai, China, 1979.
68. Kline, M. *Mathematical Thought from Ancient to Modern Times (2)*; Shanghai Science and Technology Press: Shanghai, China, 1979.
69. Kline, M. *Mathematical Thought from Ancient to Modern Times (3)*; Shanghai Science and Technology Press: Shanghai, China, 1979.
70. Kline, M. *Mathematical Thought from Ancient to Modern Times (4)*; Shanghai Science and Technology Press: Shanghai, China, 1979.
71. Solow, D. *How to Read and Do Proofs: An Introduction to Mathematical Through Process*; Science Press: Beijing, China, 1982.
72. Yan, X. Review and implication of studies on modern mathematical thinking method. *J. Math. Educ.* **2008**, *17*, 84–88. (In Chinese)
73. Shi, N. The future develpment of mathematics education. *Math. Teach* **2014**, 1–3. (In Chinese)
74. Lee, S.-Y. Mathematics learning and teaching in the school context: Reflections from cross-cultural comparisons. In *Global Prospects for Education: Development, Culture, and Schooling*; Garis, S.G., Wellman, H.M., Eds.; American Psychological Association: Washington, DC, USA, 1998; pp. 45–77.
75. Schoenfeld, A.H. Looking toward the 21st century: Challenges of educational theory and practice. *Educ. Res.* **1999**, *28*, 4–14. [CrossRef]

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).

المنارة للاستشارات

www.manaraa.com

Reproduced with permission of copyright owner. Further reproduction prohibited without permission.

المنارة للاستشارات

www.manaraa.com